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IRREGULAR TRAFFIC FLOW ON A RING ROAD†

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A model of irregular single-lane traffic flows on a ring road is proposed, with a number of fundamental differences from those traditionally examined in continuum mechanics. Even in the simplified version of the examination of single-lane traffic, the model enables a correct qualitative and quantitative description to be obtained of the conditions for ensuring maximum carrying capacity and for the occurrence and evolution of traffic jams on roads. © 2000 Elsevier Science Ltd. All rights reserved.

In the available mathematical models describing road traffic along a single-lane arterial road from the point of view of continuum mechanics [1–4], use is made of the classical of continuity equation, supplemented either with algebraic conditions expressing the empirical relationship between flow rate and speed [1, 2] or with source terms [4] or with a one-dimensional hydrodynamic equation of motion [3], which introduces into the model such properties of a compressible medium as the two-way propagation of disturbances, the absence of constraints on speeds and accelerations, etc. Therefore, transport flows, characterized by a number of specific properties such as the one-way propagation of weak disturbances, the presence of several type of wave of strong discontinuities and a constraint on speeds and accelerations, cannot be described using normal hydrodynamic models.

Two-way transport flows have also been investigated using the continuity equation alone [5].

In this paper, a traffic model is proposed that contains both the continuity equation and a differential equation of motion that takes into account the reaction of the driver to a change in the road situation and the technical characteristics of the vehicle.

1. MODEL OF THE TRAFFIC FLOW ALONG AN ARTERIAL ROAD

We will examine the one-way flow of vehicles along an arterial road with single-lane traffic. Crossroads and the presence of traffic lights will be taken into account by appropriate boundary conditions. We will introduce an Eulerian coordinate system x along the arterial road in the traffic flow direction and the time t. We will introduce the average flow density $\rho(x, t)$ as the ratio of the area of the lane occupied by vehicles to the area of the entire lane section examined

$$\rho = hln/(hL), \quad 0 \leq \rho \leq 1$$

where h is the width of the lane. L is the length of the monitored section, l is the average length of a vehicle and n is the number of vehicles on the monitored section.

We will introduce the flow speed $\upsilon(x, t)$, which can vary in the range $0 \leq \upsilon \leq \upsilon_m$, where υ_m is the speed limit. From the definitions it follows that the maximum density $\rho = 1$ corresponds to a situation where vehicles are positioned practically bumper to bumper, in this case it is natural to assume that $\upsilon = 0$, i.e. that a jam has formed on the road.

Conventionally calling

$$m = \int_0^L \rho dx$$

the "mass" concentrated on a section of length L, it is possible to write the changes in "mass" on the arterial road. For a continuous flow of vehicles, the following continuity equation will occur

$$\rho_t + (\rho\upsilon)_x = 0 \tag{1.1}$$

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In writing the equation of the change in the amount of traffic, we will take account of the fact that changes in speed are determined by the technical characteristics of the vehicle (the maximum acceleration and emergency braking) and by the reaction of the driver to a change in the road situation (a change in the density and speed of the vehicles in front). Then the equation of motion becomes

$$\frac{dv}{dt}=\begin{cases}\alpha H(\alpha)H(-\rho_x v_x), & v=0\\ \alpha H(-\rho_x v_x), & 0<v<v_m;\\ \alpha H(-\alpha)H(-\rho_x v_x), & v=v_m\end{cases}\quad \alpha=\begin{cases}\alpha^+, & \alpha_*\geqslant\alpha^+\\ \alpha_*, & -\alpha^-<\alpha_*<\alpha^+\\ -\alpha^-, & \alpha_*\leqslant-\alpha^-\end{cases} \tag{1.2}$$

$$\alpha_*=-k^2\rho^{-1}\rho_x$$

where $H(x)=\{1 \text{ when } x\geqslant 0; 0 \text{ when } x<0\}$ is the Heaviside step function, α^+ is the maximum possible acceleration of the vehicle and α^- is the deceleration of the emergency breaking; these quantities are positive and are determined by the technical characteristics of the vehicle. The parameter k, as will be seen below, characterizes the rate of propagation of small disturbances (the "speed of sound" in traffic flow); to a first approximation, we will assume that the parameter k is constant.

It can be seen from (1.2) that, in a regular traffic situation, the actions of the driver are dictated by the flow density gradient ρ_x and the density ρ itself. When the flow density increases ($\rho_x > 0$) the driver slows down ($\alpha < 0$), when the flow density decreases ($\rho_x > 0$) he accelerates ($\alpha > 0$) and in a stable situation ($\rho_x = 0$) he travels at a constant speed ($\alpha = 0$). Equation (1.2) also allows for the fact that, when there is a local increase in the flow density in front ($\rho_x > 0$), given a simultaneous increase in the speed of the vehicles in front ($v_x > 0$), the driver anticipates a subsequent reduction in density and maintains a constant speed ($\alpha = 0$). When slow-moving vehicles appear in front ($v_x < 0$), the driver does not alter his speed ($\alpha = 0$) while they are a considerable distance ahead ($\rho_x < 0$) and begins to react as they are approached ($\rho_x \geqslant 0$). Furthermore. Eq. (1.2) includes a constraint on the vehicle speed in the range $0 \leqslant v \leqslant v_m$.

An analysis of relation (1.2) indicates that the given model, unlike those examined earlier, has no direct hydrodynamic analogy and takes into account the traffic flow constraints on the speed and acceleration of individual elements and the features of the reaction of the driver to a change in the road situation.

We will examine the case of regular traffic. In this case, Eq. (1.2) takes the form

$$v_t+vv_x+k^2\rho^{-1}\rho_x=0 \tag{1.3}$$

and, together with Eq (1.1), forms a system of two quasilinear hyperbolic type equations. Its characteristics in the (x, t) plane C^+ and C^-, and the conditions along them are as follows:

$$C^\pm:\ dx/dt=v\pm k,\quad \rho dv=\mp k d\rho \tag{1.4}$$

The characteristics C^+ and C^- carry information on any change in the road situation in the flow direction and in the opposite direction respectively.

A specific feature of traffic flows and existing systems of traffic organization is that the propagation of information is one way in the upstream direction. For waves propagating to the left, in the upstream direction, in the case of the flow with constant parameters, the Riemann integral occurs

$$-v_0=k\ln(\rho_0/\rho) \tag{1.5}$$

We will will assume that, for zero flow speed, the vehicles are standing on the arterial road bumper to bumper and their density is a maximum ($\rho_0 = 1$ when $v_0 = 0$). Then, from integral (1.5), taking account of the constraints on the maximum speed ($v \leqslant v_m$), it is possible to obtain the following relation for the flow speed

$$v(\rho)=\begin{cases}-k\ln\rho, & e^{-v_m/k}<\rho\leqslant 1\\ v_m, & 0<\rho\leqslant e^{-v_m/k}\end{cases} \tag{1.6}$$

The graph of the function $q(\rho) = \rho v$, defining the traffic capacity of a single-lane road (Fig. 1), indicates that maximum traffic capacity (MTC) is achieved for a density $\rho = 1/e$ and a speed $v = k$.

To estimate the rate of propagation of disturbances k, we will make the following assumptions

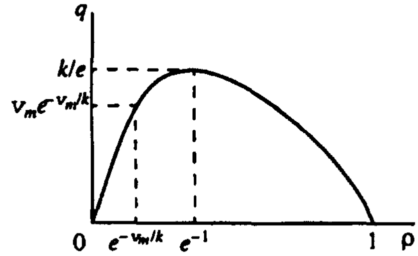

Fig. 1.

concerning the properties of the flow. Suppose, starting from a state of rest ($\upsilon = 0$; $\rho = 1$) and accelerating to υ_m, the flow reaches its maximum permissible density ρ_* guaranteeing traffic safety. The safe density will be taken to mean that for which the distances between vehicles are no shorter than the braking distance $X(\upsilon)$. Then the maximum permissible density at speed υ_m will be defined by the equality

$$\rho_* = (1 + X(\upsilon_m)/l)^{-1}$$

where l is the characteristic length of the vehicle. On the other hand, from the Riemann integral for on acceleration wave (1.5), the density that can be achieved in it is determined by the final traffic speed and by the rate of propagation of disturbances k

$$\rho_* = e^{-\upsilon_m/k}$$

Then the rate of propagation of disturbances k is defined by the formula

$$k = \upsilon_m / \ln(1 + X(\upsilon_m)/l)$$

At $\upsilon_m = 80$ km/h, the braking distance of a VAZ-type vehicle is 45 m, which, for an average vehicle length of 5 m, gives a rate of propagation of disturbances $k = 35$ km/h, and estimates of the maximum acceleration α^+ and braking deceleration α^- [formula (1.2)] for vehicles of this class $\alpha^+ = 1.63$ m/s^2 and $\alpha^- = 5.5$ m/s^2.

An analysis was made in [6] of experimental data from traffic observations in the Lincoln Tunnel in New York, showing that the dependence of the speed on the flow density is approximated quite well by an expression of type (1.6) . Here, the coefficient k, determined from experimental data, amounted to $k = 17.2$ miles/h $\cong 28$ km/h. The lower k value obtained in the experiment is probably due to the fact that the real flow density is always slightly lower than the maximum permissible density ($\rho < \rho_*$) for the purposes of increasing traffic safety.

Experiments in the Lincoln Tunnel [6] showed that the MTC of the tunnel of 1430 vehicles per hour was achieved for a density of 83 vehicles per mile. Here, the maximum dimensional density $\hat{\rho}_m$ was 228 vehicles per mile. This indicates that the density with the MTC is $\rho_m = 83/228 \approx 0.36$, which is in good agreement with the theoretically determined value $\rho_m = 1/e \approx 0.37$. The formula for the MTC in dimensional variables $\hat{q}_m = k\hat{\rho}_m$ gives $\hat{q}_m = 17.3 \times 83 = 1436$ vehicles per hour, which again is in good agreement with experiment.

Experiments on the Merritt Parkway [3] showed that a MTC of 1300 vehicles per hour was achieved with a density $\hat{\rho}_m = 80$ vehicles per mile with a maximum density in a state of rest of 215 vehicles per mile, which gives $\rho_m \approx 0.37 \approx 1/e$, again in good agreement with theoretical conclusions.

2. ANALYSIS OF SOME SOLUTIONS OF THE SYSTEM OF EQUATIONS

We will examine simple waves propagating to the left, in the upstream direction. For these waves, the characteristics C^-: $dx/dt = \upsilon - k$ will be straight lines, while the characteristics C^+ will be curvilinear. The Riemann integral (1.5) holds over the entire region of the simple wave. Changes in the rate of propagation of small disturbances in relation to the flow speed and density are respectively equal to

$$d(\upsilon - k)/d\upsilon = 1, \quad d(\upsilon - k)/d\rho = -k/\rho$$

from which it follows that the slope of the characteristics along the trajectory increases when the speed increases and the density decreases and decreases when the speed decreases and the density increases. Thus, acceleration waves are characterized by fan-like diverging characteristics, while deceleration waves braking are characterized by converging curves. In the latter case, intersection of the curves is possible, leading to the emergence of a gradient catastrophe and a region of multiple valued solution. Therefore, it is necessary to supplement the model by taking into account the possible emergence of strong discontinuities.

The conditions on the surface of a strong discontinuity for the model examined, obtained by writing Eqs (1.1) and (1.3) in integral form, are as follows:

$$\rho_1(D - v_1) = \rho_2(D - v_2) \tag{2.1}$$

$$\rho_1((D - v_1)^2 + k^2) = \rho_2((D - v_2)^2 + k^2) \tag{2.2}$$

We will consider possible solutions of problems of the decay of discontinuities under the initial conditions. Suppose that, when $x = x_0$, there is a discontinuity

$$v = v_1,\ \rho = \rho_1 \text{ when } x < x_0;\ v = v_0,\ \rho = \rho_0 \text{ when } x > x_0.$$

Then, when $v_1 < v_0$, a centred acceleration wave propagates in the upstream in direction, which in the downstream direction, with the speed of the flow, a contact discontinuity propagates (Fig. 2a). In a continuous acceleration wave, the speed increases from v_1 to v_0, and the density decreases to a value $\rho_2 = \rho_1 \exp[(v_1 - v_0)/k]$. On the contact surface k there is a discontinuity in density. Note that a solution of the centred-wave type has a singularity when $t = 0$ and $x = x_0$ and holds when $t > v_m/\alpha^+$.

When $v_1 > v_0$, a deceleration wave braking in the form of a strong discontinuity propagates in the upstream direction against the flow. Two types of solution are possible. For a relatively small speed difference $(v_1 - v_0)$ the solution is shown in Fig. 2(b).

The wave speed D and the density of the braked flow ρ_2 are determined from relations (2.1) and (2.2)

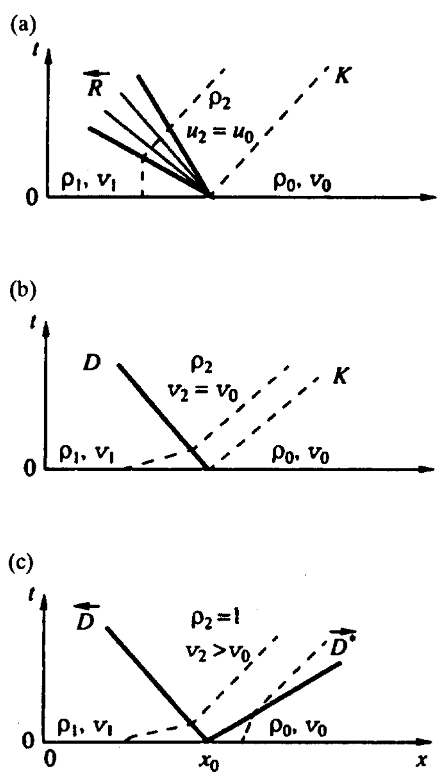

Fig. 2.

$$D = \frac{1}{2}(v_1 + v_0 - \Delta), \quad \rho_2 = \frac{\rho_1}{4k^2}(v_1 - v_0 + \Delta)^2 \tag{2.3}$$

$$\Delta = \sqrt{(v_1 - v_0)^2 + 4k^2}$$

This solution holds so long as the condition

$$\Delta < 2k\rho_1^{-1/2} - (v_1 - v_0) \tag{2.4}$$

which ensures that the density ρ_2 does exceed its maximum value, is satisfied. When this condition is not satisfied, the solution has the form shown in Fig. 2(c).

A deceleration wave propagates in the upstream direction the flow at a speed

$$D = v_1 - k\rho_1^{-1/2} \tag{2.5}$$

which, under certain conditions, may be carried downstream to the right by the flow. Behind the wave a steady speed

$$v_2 = v_1 - k\rho_1^{-1/2}(1 - \rho_1) \tag{2.6}$$

A packing wave propagates is established in the downstream direction at a rate.

$$D_* = (v_1 - v_0)/(1 - \rho_0) + v_0 - k\rho_1^{-1/2} > v_0. \tag{2.7}$$

The packing wave in this case is referred to the wave propagating in the downstream direction, behind which the medium density is equal to the final maximum possible value and cannot increase any further. This distinguishes the waves examined fundamentally from gas-dynamic strong discontinuities, imposing different conditions on the wave propagation velocity. The condition $D_* > v_0$ arises if condition (2.4) is not satisfied, and hence

$$v_1 - v_0 > k\rho_1^{-1/2}(1 - \rho_0)$$

Note that the speed of the packing wave (2.7) was determined using the condition for the conservation of mass flow, which has the form of (2.1). The condition for the conservation of the momentum flux for the packing wave has a form different from (2.2) and is used to determine the unknown stresses that arise when physical contact is established between the system elements.

3. SOLUTION OF A MODEL PROBLEM

We will construct the solution of a model problem of the dynamics of traffic on a one-way ring road. Suppose there is a traffic lane of length L where the boundary conditions when $x = 0$ are the values of the solution when $x = L$. Suppose that, at the initial instant of time, stationary vehicles are concentrated in a section $x_1 < x < x_2$ and have a density $\rho_0 < l$ ($v_0 = 0$). Suppose the values of the characteristic speeds for the system examined are $v_m = 3V$ and $k = 2V$, which allows the existence of "subsonic" and "supersonic" traffic modes. We will also assume that $\alpha^+ L \gg V^2$ and $\alpha^- L \gg V^2$. The traffic wave pattern is shown in Fig. 3.

When the discontinuity at the point $x = x_2$ decays, a centred acceleration wave is formed, in which the vehicles accelerate to the speed v_m and then travel at constant speed. The main characteristic of the acceleration wave ($x = x_2 - kt$) propagates to the left, in the upstream direction. The "tail" of the wave also propagates in the upstream direction, but the latter characteristic $[x = x_2 + (v_m - k)t]$ is taken away to the right by the flow. The main flow of vehicles travels uniformly: $x = x_2 + v_m t$. When the condition $x_2 - x_1 > L/(1 + v_m/k)$, which occurs in the given problem ($x_2 - x_1 > 2L/5$), is satisfied, the leading vehicles after a certain time (at the point $x = x_1$) encounter stationary vehicles still not reached by the acceleration wave. The flow slows down until it arrests entirely. A acceleration wave $x = x_1 - Dt$ propagates in the upstream direction at a speed $D = -V$. The braked flow remains at rest until the leading characteristic of the acceleration wave reaches it, after which the flow acceleration again occurs (but more smoothly). A typical example of the trajectory of motion is shown in Fig. 3 by the dashed line.

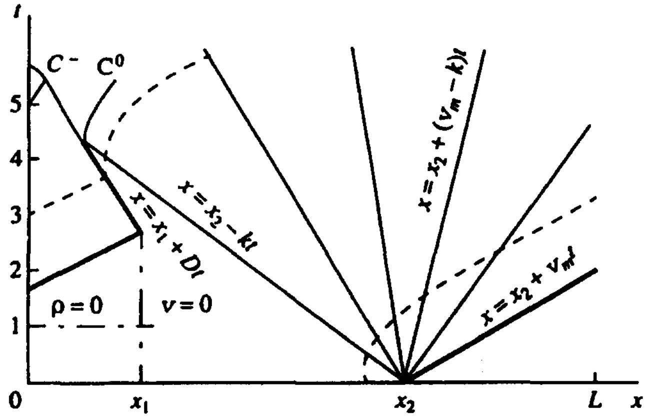

Fig. 3.

After a certain time, the leading characteristic of the acceleration wave ($x = x_2 - 2Vt$) catches up with the deceleration wave ($x = x_1 - Vt$). At the point where they interact the expanding region where a general solution of the system exists has its origin. This region is bounded by the characteristics C^- and C^0.

The density and speed distribution profiles for three successive instants of time are presented in Fig. 4. It can be seen that the traffic speed (the dashed lines) increases linearly in the acceleration wave from 0 to v_m. However, the flow density decreases exponentially from ρ_0 to $\rho_0 e^{-3/2}$. In the deceleration wave the density then increases to $\rho = 4\rho_0 e^{-3/2} < \rho_0$. Graphs of the density and speed distribution for all instants of time pass through the same points when $x = x_2$; $\rho = \rho_0 e^{-1}$ and $v = k$ for all values of t.

4. NUMERICAL SOLUTION OF THE PROBLEM OF IRREGULAR TRAFFIC ON A RING ROAD

The system of equations obtained was solved numerically by the flow correction method [7] with second order of accuracy. Here, the initial and boundary conditions corresponded to the example examined in Section 3. Accurate analytical solutions obtained for initial instants of time $t \leq 3.5$ (Fig. 4) were used as a test for the numerical algorithm selected. An analysis of the results of the numerical solution – the density distribution (Fig. 5), the speed distribution (Fig. 6) and the capacity distribution (Fig. 7) for the corresponding time – indicates that it possesses the accuracy required. The inevitable "blurring" of the leading edge in the numerical implementation leads to the appearance of a "forerunner" of low density, also travelling at maximum speed, but this does not spoil the traffic pattern obtained as a whole.

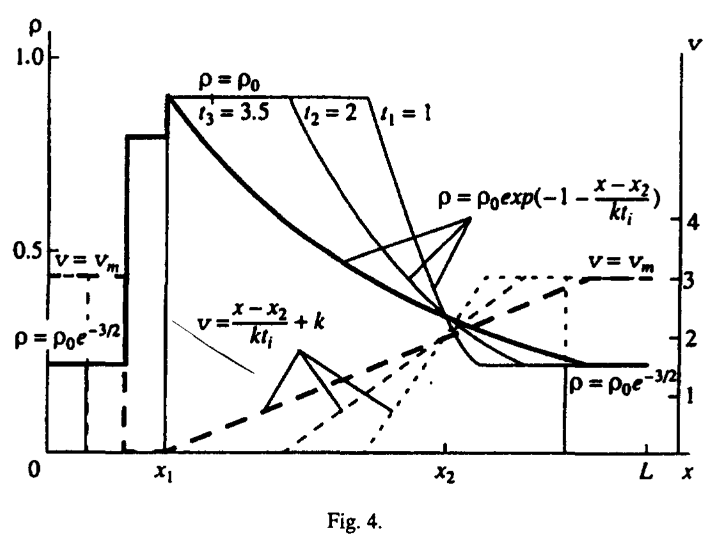

Fig. 4.

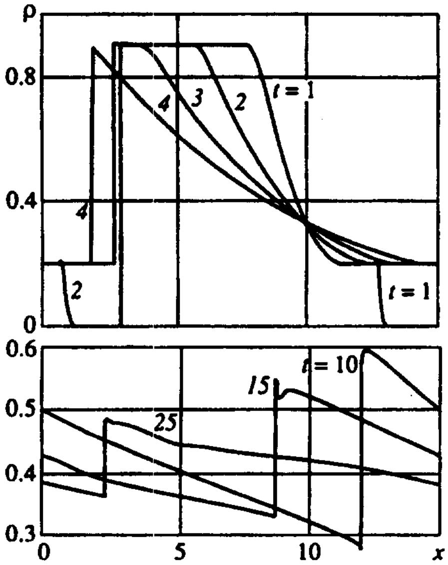

Fig. 5.

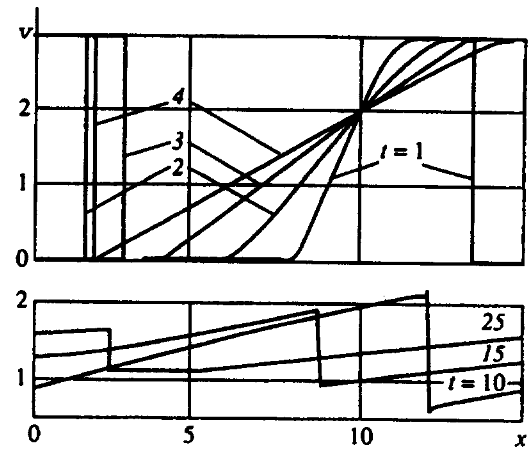

Fig. 6.

Figure 5 gives density profiles for successive instants of time. It can be seen that, after the accelerations wave interacts with the deceleration wave, the latter begins to weaken but continues to propagate in the upstream direction, reaches $x = 0$ and then appears at $x = L$ (the lower part of Fig. 5).

An analysis of the traffic carrying capacity profiles (Fig. 7) shows the MTC is reached when $x = x_2$, when the flow speed is equal to the local speed of sound $\upsilon = k$, and the density $\rho = 1/e$, which is consistent with the theoretical conclusions.

The results of the numerical solution indicate that a traffic jam arises as time passes; on a single-lane ring road. This jam moves in the flow, upstream directions. Within the jam there is a drop in speed (but not a complete halt) and an increase in density, on the whole leading to a small reduction in carrying capacity. As time passes the amplitudes of the change in the defining parameters in the traffic jam decrease.

On entering such a jam, vehicles reduce their speed sharply and then smoothly accelerate to their former speed. Something similar to a jam may be encountered at considerable distances from the point of the initial congestion.

The example considered is characterized by a fairly high average traffic density on the ring road. Therefore steady traffic on the road is characterized by the average speed, which is considerably lower than the speed limit on the given road: $\upsilon_m = 3V$. Here, the carrying capacity q established on the road (Fig. 7) is near optimum.

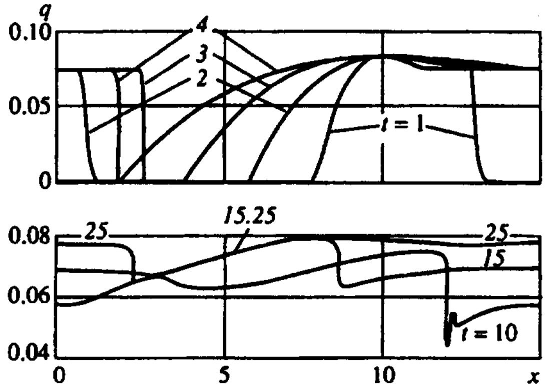

Fig. 7.

5. CONCLUSIONS

A comparison of the model proposed with models which exist in the literature indicates that the proposed model takes account of constraints on the speed and acceleration of individual elements of the traffic flow, and also the technical characteristics of the vehicles and features of the reaction of the driver to a change in the road situation. By virtue of this, the problem has no direct hydrodynamic analogy. The medium examined (traffic flows) has a number of differences from those traditionally examined in continuum mechanics: the existence of finite constraints on speeds, accelerations (positive and negative) and density, one-way propagation of weak disturbances and the possibility of the existence of different waves of strong discontinuity.

The model developed enclose a correct qualitative and quantitative description to be given of such characteristics and of the emergence and evolution of traffic jams on arterial roads.

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